

## 7. ELECTROMAGNETIC RELATIONS

Quantity	Gaussian CGS	SI
Conversion factors: Charge: Potential: Magnetic field:	$2.997\,924\,58 \times 10^9$ esu $(1/299.792\,458)$ statvolt (ergs/esu) $10^4$ gauss = $10^4$ dyne/esu	$= 1\text{ C} = 1\text{ A s}$ $= 1\text{ V} = 1\text{ J C}^{-1}$ $= 1\text{ T} = 1\text{ N A}^{-1}\text{m}^{-1}$
Lorentz force:	$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
Maxwell equations:	$\nabla \cdot \mathbf{D} = 4\pi\rho$ $\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J}$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \cdot \mathbf{D} = \rho$ $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$
Constitutive relations:	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ , $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ , $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
Linear media: Permittivity of free space: Permeability of free space:	$\mathbf{D} = \epsilon\mathbf{E}$ , $\mathbf{H} = \mathbf{B}/\mu$ 1 1	$\mathbf{D} = \epsilon\mathbf{E}$ , $\mathbf{H} = \mathbf{B}/\mu$ $\epsilon_0 = 8.854\,187 \dots \times 10^{-12}\text{ F m}^{-1}$ $\mu_0 = 4\pi \times 10^{-7}\text{ N A}^{-2}$
Fields from potentials:	$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \nabla \times \mathbf{A}$	$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \nabla \times \mathbf{A}$
Static potentials: (coulomb gauge)	$V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$ $\mathbf{A} = \frac{1}{c} \oint \frac{I d\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$	$V = \frac{1}{4\pi\epsilon_0} \sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$ $\mathbf{A} = \frac{\mu_0}{4\pi} \oint \frac{I d\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$
Relativistic transformations: ( $\mathbf{v}$ is the velocity of the primed frame as seen in the unprimed frame)	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$ $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \frac{1}{c}\mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$ $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c}\mathbf{v} \times \mathbf{E})$	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$ $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$ $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E})$
$\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7}\text{ N A}^{-2} = 8.987\,55 \dots \times 10^9\text{ m F}^{-1}$ ; $\frac{\mu_0}{4\pi} = 10^{-7}\text{ N A}^{-2}$ ; $c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.997\,924\,58 \times 10^8\text{ m s}^{-1}$		

### 7.1. Impedances (SI units)

$\rho$  = resistivity at room temperature in  $10^{-8} \Omega \text{ m}$ :  
 $\sim 1.7$  for Cu     $\sim 5.5$  for W  
 $\sim 2.4$  for Au     $\sim 73$  for SS 304  
 $\sim 2.8$  for Al     $\sim 100$  for Nichrome  
 (Al alloys may have double the Al value.)

For alternating currents, instantaneous current  $I$ , voltage  $V$ , angular frequency  $\omega$ :

$$V = V_0 e^{j\omega t} = ZI . \quad (7.1)$$

Impedance of self-inductance  $L$ :  $Z = j\omega L$ .

Impedance of capacitance  $C$ :  $Z = 1/j\omega C$ .

Impedance of free space:  $Z = \sqrt{\mu_0/\epsilon_0} = 376.7 \Omega$ .

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j)\rho}{\delta}, \quad \text{where } \delta = \text{skin depth}; \quad (7.2)$$

$$\delta = \sqrt{\frac{\rho}{\pi\nu\mu}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu(\text{Hz})}} \quad \text{for Cu} . \quad (7.3)$$

### 7.2. Capacitance $\hat{C}$ and inductance $\hat{L}$ per unit length (SI units) [negligible skin depth]

Flat rectangular plates of width  $w$ , separated by  $d \ll w$  with linear medium  $(\epsilon, \mu)$  between:

$$\hat{C} = \epsilon \frac{w}{d}; \quad \hat{L} = \mu \frac{d}{w}; \quad (7.4)$$

$$\epsilon/\epsilon_0 = 2 \text{ to } 6 \text{ for plastics; } 4 \text{ to } 8 \text{ for porcelain, glasses;} \quad (7.5)$$

$$\mu/\mu_0 \approx 1 . \quad (7.6)$$

Coaxial cable of inner radius  $r_1$ , outer radius  $r_2$ :

$$\hat{C} = \frac{2\pi\epsilon}{\ln(r_2/r_1)}; \quad \hat{L} = \frac{\mu}{2\pi} \ln(r_2/r_1) . \quad (7.7)$$

Transmission lines (no loss):

$$\text{Impedance: } Z = \sqrt{\hat{L}/\hat{C}} . \quad (7.8)$$

$$\text{Velocity: } v = 1/\sqrt{\hat{L}\hat{C}} = 1/\sqrt{\mu\epsilon} . \quad (7.9)$$

### 7.3. Synchrotron radiation (CGS units)

For a particle of charge  $e$ , velocity  $v = \beta c$ , and energy  $E = \gamma mc^2$ , traveling in a circular orbit of radius  $R$ , the classical energy loss per revolution  $\delta E$  is

$$\delta E = \frac{4\pi}{3} \frac{e^2}{R} \beta^3 \gamma^4 . \quad (7.10)$$

For high-energy electrons or positrons ( $\beta \approx 1$ ), this becomes

$$\delta E \text{ (in MeV)} \approx 0.0885 [E(\text{in GeV})]^4 / R(\text{in m}) . \quad (7.11)$$

For  $\gamma \gg 1$ , the energy radiated per revolution into the photon energy interval  $d(\hbar\omega)$  is

$$dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(\hbar\omega) , \quad (7.12)$$

where  $\alpha = e^2/\hbar c$  is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \quad (7.13)$$

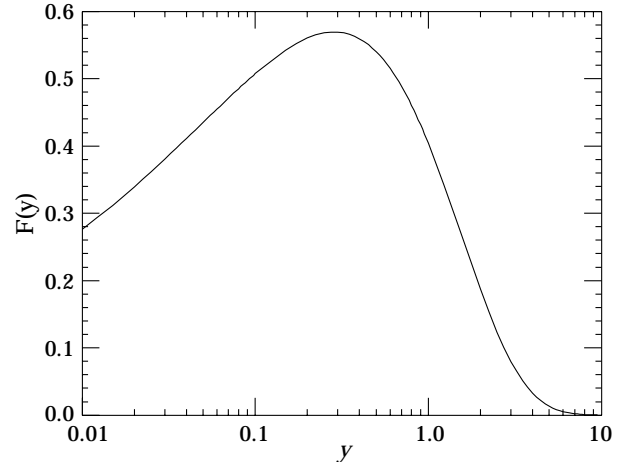
is the critical frequency. The normalized function  $F(y)$  is

$$F(y) = \frac{9}{8\pi} \sqrt{3} y \int_y^\infty K_{5/3}(x) dx , \quad (7.14)$$

where  $K_{5/3}(x)$  is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c \text{ (in keV)} \approx 2.22 [E(\text{in GeV})]^3 / R(\text{in m}) . \quad (7.15)$$

Fig. 7.1 shows  $F(y)$  over the important range of  $y$ .



**Figure 7.1:** The normalized synchrotron radiation spectrum  $F(y)$ .

For  $\gamma \gg 1$  and  $\omega \ll \omega_c$ ,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha (\omega R/c)^{1/3} , \quad (7.16)$$

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c ,$$

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c} \left[1 + \frac{55}{72} \frac{\omega_c}{\omega} + \dots\right] . \quad (7.17)$$

The radiation is confined to angles  $\lesssim 1/\gamma$  relative to the instantaneous direction of motion. The mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}} \alpha \gamma , \quad (7.18)$$

and the mean energy per photon is

$$\langle \hbar\omega \rangle = \frac{8}{15\sqrt{3}} \hbar\omega_c . \quad (7.19)$$

When  $\langle \hbar\omega \rangle \gtrsim O(E)$ , quantum corrections are important.

See J.D. Jackson, *Classical Electrodynamics*, 2<sup>nd</sup> edition (John Wiley & Sons, New York, 1975) for more formulae and details. In his book, Jackson uses a definition of  $\omega_c$  that is twice as large as the customary one given above.