DALITZ PLOT ANALYSIS FORMALISM

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Introduction: Weak nonleptonic decays of D and B mesons are expected to proceed dominantly through resonant two-body decays [1]; see Ref. 2 for a review of resonance phenomenology. The amplitudes are typically calculated with the Dalitz-plot analysis technique [3], which uses the minimum number of independent observable quantities. For three-body decays of a spin-0 particle to all pseudo-scalar final states, such as D or $B \to abc$, the decay rate [4] is

$$\Gamma = \frac{1}{(2\pi)^3 32\sqrt{s^3}} \left|\mathcal{M}\right|^2 dm^2_{ab}\, dm^2_{bc}\,, \tag{1}$$

where m_{ij} is the invariant mass of particles i and j. Here the prefactor contains all kinematic factors, while $\left|\mathcal{M}\right|^2$ contains the dynamics. The scatter plot in m^2_{ab} versus m^2_{bc} is the Dalitz plot. If $\left|\mathcal{M}\right|^2$ is constant, the kinematically allowed region of the plot will be populated uniformly with events. Any variation in the population over the Dalitz plot is due to dynamic rather than kinematic effects. It is straightforward to extend the formalism beyond three-body final states.

Formalism: The amplitude for the process $R \to rc, r \to ab$ where R is a D or B meson, r is an intermediate resonance, and a, b, c are pseudo-scalars, is given by

$$\mathcal{M}_r(J,L,l,m_{ab},m_{bc}) = \sum_\lambda \langle ab|r_\lambda\rangle\, T_r(m_{ab})\, \langle cr_\lambda|R_J\rangle \tag{2}$$

$$= Z(J,L,l,\vec{p},\vec{q}) B^R_L(|\vec{p}|) B^r_L(|\vec{q}|) T_r(m_{ab})\,.$$

The sum is over the helicity states λ of r; J is the total angular momentum of R (for D and B decays, $J = 0$); L is the orbital angular momentum between r and c; l is the orbital angular momentum between a and b; (the spin of r); $\vec{p}$ and $\vec{q}$ are the momenta of c and of a in the r rest frame; Z describes the angular distribution of the final-state particles; B^R_L and B^r_L are the barrier factors for the production of rc and of ab; and T_r is the dynamical function describing the resonance r. T_r is a phenomenological object, with the resonances modeled often by a Breit-Wigner form, although some more recent analyses use a K-matrix formalism [5–7] with the P-vector approximation [8] to describe the $\pi\pi$ S-wave.

The nonresonant (NR) contribution to $D \to abc$ is parametrized as constant (S-wave), with no variation in magnitude or phase across the Dalitz plot. The available phase space is much greater for B decays than for D decays, and the nonresonant contribution to $B \to abc$ requires a more sophisticated parametrization. Experimentally, several parametrizations have been used [9,10]. Differences in the parametrizations of the NR contributions, and in Z, B_L, and T_r, as well as in the set of resonances r, complicate the comparison of results from different experiments.

Angular distribution Z: The tensor or Zemach formalism [11,12] and the helicity formalism [13,12] yield identical descriptions of the angular distributions for the decay process $R \to rc, r \to ab$ when a, b and c all have spin 0. The angular distributions for $L = 0, 1$, and 2 are given in Table 1. For a derivation of the expressions, see, *e.g.*, Ref. 12. For final-state particles with non-zero spin (*e.g.*, radiative decays), the helicity formalism is required.

Table 1: Angular distributions for $L = 0, 1, 2$ for the decay process $R \to rc, r \to ab$ when a, b and c all have spin 0. Here θ is the angle between particles a and c in the rest frame of resonance r, $\sqrt{1+\zeta^2} = E_r/m_{ab}$ is a relativistic correction, where $E_r = (m_R^2 + m_{ab}^2 - m_c^2)/2m_R$ is the energy of resonance r in the rest frame of R.

$J \to L + l$	Angular distribution
$0 \to 0+0$	uniform
$0 \to 1+1$	$(1+\zeta^2)\cos^2\theta$
$0 \to 2+2$	$\left(\zeta^2+\frac{3}{2}\right)^2 (\cos^2\theta - 1/3)^2$

Barrier Factor B_L: The maximum angular momentum L in a strong decay is limited by the linear momentum q — the relative momentum of the decay particles in the center of mass frame of the decaying resonance. Decay particles moving slowly with an impact parameter (meson radius) d of order 1 fm have difficulty generating sufficient angular momentum to conserve the spin of the resonance. The Blatt-Weisskopf [14,15] functions B_L, given in Table 2, weight the reaction amplitudes to account for this spin-dependent effect. These functions are normalized to give $B_L = 1$ for $z = (|q|\,d)^2 = 1$. Another common formulation, B'_L, also in Table 2, is normalized to give $B'_L = 1$ for $z = z_0 = (|q_0|\,d)^2$ where q_0 is the value of q when $m_{ab} = m_r$. An important difference between the B_L and the B'_L is that the former include explictly the centrifugal barrier, while it is to be moved to the dynamical functions in the case of B'_L.

Table 2: Blatt-Weisskopf barrier factors weight the reaction amplitudes to account for spin-dependent effects (c.f. Sec. VIII.5 of Ref. 14). Two formulations with different normalization conditions (described in text) are shown. B_L is commonly used in Dalitz plot analyses; B'_L is commonly used with the helicity formalism.

L	$B_L(q)$	$B'_L(q, q_0)$
0	1	1
1	$\sqrt{\dfrac{2z}{1+z}}$	$\sqrt{\dfrac{1+z_0}{1+z}}$
2	$\sqrt{\dfrac{13z^2}{(z-3)^2+9z}}$	$\sqrt{\dfrac{(z_0-3)^2+9z_0}{(z-3)^2+9z}}$
where $z = (\lvert q\rvert\,d)^2$ and $z_0 = (\lvert q_0\rvert\,d)^2$		

Dynamical Function T_r: The dynamical function T_r is derived from the S-matrix formalism [5]. In general, the amplitude that a final state f couples to an initial state i is $S_{fi} = \langle f|S|i\rangle$, where the scattering operator S is unitary:

$SS^\dagger = S^\dagger S = I$. The Lorentz-invariant transition operator $\hat{T}$ is defined by separating the probability that $f = i$, yielding

$$S = I + 2iT = I + 2i\,\{\rho\}^{1/2}\,\hat{T}\,\{\rho\}^{1/2}\,, \tag{3}$$

where I is the identity operator and ρ is the diagonal phase-space matrix. If channel i denotes the two–body state ab, then

$$\rho_{ii} = \rho_i = \frac{2q_i}{m_{ab}}\,\theta\left[m_{ab} - (m_a + m_b)\right]\,, \tag{4}$$

where m_{ab} is the invariant mass of the system;

$$q_i = \frac{1}{2m_{ab}}\sqrt{(m_{ab}^2 - (m_a + m_b)^2)(m_{ab}^2 - (m_a - m_b)^2)} \tag{5}$$

is the momentum of a in the r rest frame, and $\theta[...]$ is the step function. In the single-channel case, unitarity allows one to express S through a single parameter, $S = e^{2i\delta}$, and

$$\hat{T} = \frac{1}{\rho}\,e^{i\delta}\sin\delta. \tag{6}$$

There are three common formulations of the dynamical function. The Breit-Wigner form—the first term in a Taylor expansion about a T-matrix pole—is the simplest. The K-matrix formalism [5] is more general (allowing more than one T-matrix pole and coupled channels while preserving unitarity). The Flatté distribution [16] is used to parametrize resonances near threshold, located at $s = (m_a + m_b)^2$, and is equivalent to a one-pole, two-channel K-matrix.

Breit-Wigner Form

The common formulation of a Breit-Wigner resonance decaying to spin-0 particles a and b is

$$T_r(m_{ab}) \propto \frac{1}{m_r^2 - m_{ab}^2 - im_r\Gamma_{ab}(m_{ab})}\,. \tag{7}$$

A standard formulation for the "mass-dependent" width Γ_{ab} reads

$$\Gamma_{ab}(m_{ab}) = \sum_i \Gamma_i^r \left(\frac{q_i}{q_r}\right)^{2L_i+1}\left(\frac{m_r}{m_{ab}}\right) B'_{L_i}(q_i, q_0)^2\,, \tag{8}$$

where q_i, L_i, Γ_i^r and $B'_{L_i}(q_i, q_0)$ are the momentum and angular momentum of the decay products, the partial width and Blatt-Weisskopf barrier factor (see Table 2) for the decay of resonance r into channel i, respectively. A Breit-Wigner parametrization best describes isolated, non-overlapping resonances far from the threshold of additional decay channels. For the $\rho(770)$ and $\rho(1450)$ a more complex parametrization suggested by Gounaris-Sakarai [17] is often used [18-22]. Unitarity is violated when the dynamical function is parametrized as the sum of two or more overlapping Breit-Wigners — see the discussion below Eq. (13). The proximity of a threshold to a resonance distorts the line shape from a simple Breit-Wigner. Here the Flatté formula provides a better description and is discussed below.

K-matrix Formulation

The T matrix can be written as

$$\hat{T} = (I - i\hat{K}\rho)^{-1}\hat{K}, \tag{9}$$

where $\hat{K}$ is the Lorentz-invariant K-matrix describing the scattering process and ρ is the phase-space factor defined below Eq. (3). Resonances appear as poles in the K-matrix:

$$\hat{K}_{ij} = \sum_{\alpha} \frac{\sqrt{m_\alpha \Gamma_{\alpha i}(m) m_\alpha \Gamma_{\alpha j}(m)}}{(m_\alpha^2 - m^2)\sqrt{\rho_i \rho_j}}. \tag{10}$$

The K-matrix is real by construction, and so the associated T-matrix respects unitarity. However, in the given form it has the wrong analytic structure. To improve it, some authors use the analytic continuation for the momentum q_i, defined in Eq. (5), to below-threshold values, where for $m_a \neq m_b$ the phase space factor needs to be modified to avoid false singularities (see, *e.g.*, Ref. 7, Sec. 2.1). For further improvements see below.

For a single pole in a single channel, $K = \rho\hat{K}$ is

$$K = \frac{m_0\,\Gamma(m)}{m_0^2 - m^2} \tag{11}$$

and

$$T = K(1 - iK)^{-1} = \frac{m_0 \Gamma(m)}{m_0^2 - m^2 - i m_0 \Gamma(m)}\,, \tag{12}$$

which is the relativistic Breit-Wigner formula. For two poles in a single channel, K is

$$K = \frac{m_\alpha \Gamma_\alpha(m)}{m_\alpha^2 - m^2} + \frac{m_\beta \Gamma_\beta(m)}{m_\beta^2 - m^2}\,. \tag{13}$$

If m_α and m_β are far apart relative to the widths, the T matrix is approximately the sum of two Breit-Wigners, $T(K_\alpha + K_\beta) \approx T(K_\alpha) + T(K_\beta)$, each of the form of Eq. (12). This approximation is not valid for two nearby resonances, for it violates unitarity. For example, for $m = m_\alpha$ the full, unitary K-matrix expression gives Im(T)=1, while the imaginary part of $T(K_\alpha) + T(K_\beta)$ is $1 + (m_\beta\Gamma_\beta)^2/[(m_\beta^2 - m_\alpha^2)^2 + (m_\beta\Gamma_\beta)^2]$.

This formulation, which applies to S-channel production in two-body scattering, $ab \to cd$, can be generalized to describe the production of resonances in processes such as the decay of charm mesons. The key assumption here is that the two-body system described by the K-matrix does *not* interact with the rest of the final state [8]. The validity of this assumption varies with the production process and is appropriate for reactions such as $\pi^- p \to \pi^0\pi^0 n$ in the several-GeV regime, and for semileptonic decays such as $D \to K\pi\ell\nu$. The assumption may be of limited validity for production processes such as $p\overline{p} \to \pi\pi\pi$, $D \to \pi\pi\pi$, $D \to K\pi\pi$ and $J/\psi \to \omega\pi\pi$. In the last two cases, additional three–body rescatterings were found to be relevant. In the J/ψ decays, they appeared where the two–body amplitudes were very small [23]; in the D decays, they were shown to lead to a significant difference between the $K\pi$ scattering phase and the phase extracted from the production process [24]. If three–body interactions are neglected, the two-body Lorentz-invariant amplitude, $\hat{F}$, is given by

$$\hat{F}_i = (I - i\hat{K}\rho)^{-1}_{ij}\hat{P}_j = (\hat{T}\hat{K}^{-1})_{ij}\hat{P}_j\,, \tag{14}$$

where P is the production vector that parametrizes the resonance production in the open channels.

For the $\pi\pi$ S-wave, a common formulation of the K-matrix [7,20,21,25] is

$$K_{ij}(s) = \left[\sum_\alpha \left(\frac{g_i^{(\alpha)} g_j^{(\alpha)}}{m_\alpha^2 - s}\right) + f_{ij}^{sc}\frac{1 + s_0^{sc}}{s + s_0^{sc}}\right]\left[\frac{(s - s_A)}{(s + s_{A0})}\right]\,. \tag{15}$$

The factor $g_i^{(\alpha)}$ is the real coupling constant of the K-matrix pole m_α to meson channel i; the parameters f_{ij}^{sc} and s_0^{sc} describe a smooth part of the K-matrix elements; the second factor in square brackets, with $s_A \sim (0.1-0.5)m_\pi^2$ contains the Adler zero and at the same time suppresses a false kinematical singularity; *e.g.*, in Ref. 25, $s_{A0} = 0.15$ GeV^2 and $s_A = 0.5m_\pi^2$ were used. The number 1 has units GeV^2.

The production vector, with $i = 1$ denoting $\pi\pi$, is

$$P_j(s) = \left[\sum_\alpha (\frac{\beta_\alpha g_j^{(\alpha)}}{m_\alpha^2 - s}) + f_{1j}^{pr} \frac{1+s_0^{pr}}{s+s_0^{pr}}\right] , \qquad (16)$$

where the free parameters of the Dalitz-plot fit are the complex production couplings β_α and the production-vector background parameters f_{1j}^{pr} and s_0^{pr}. All other parameters are fixed by scattering experiments. Ref. 6 describes the $\pi\pi$ scattering data with a 4-pole, 2-channel ($\pi\pi$, $K\bar{K}$) model, while Ref. 7 describes the scattering data with a 5-pole, 5-channel ($\pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta'\eta'$ and 4π) model. The former has been implemented by CLEO [26] and the latter by FOCUS [21] and BABAR [20]. In both cases, only the $\pi\pi$ channel was analyzed. A more complete coupled-channel analysis would simultaneously fit all final states accessible by rescattering.

Flatté Formalism

The Flatté formulation is used when a second channel opens close to a resonance. This situation occurs in the $\pi\pi$ S-wave where the $f_0(980)$ is near the $K\overline{K}$ threshold, and in the $\pi\eta$ channel where the $a_0(980)$ also lies near the $K\overline{K}$ threshold. The T-matrix is parameterized as

$$\hat{T}(m_{ab})_{ij} = \frac{g_i g_j}{m_r^2 - m_{ab}^2 - i(\rho_1 g_1^2 + \rho_2 g_2^2)} , \qquad (17)$$

where $\rho_1 g_1^2 + \rho_2 g_2^2 = m_0 \Gamma_r$, when the phase spaces are evaluated at the resonance mass. For the $a_0(980)$ resonance, the relevant coupling constants are $g_1 = g_{\pi\eta}$ and $g_2 = g_{KK}$, and the phase space terms are $\rho_1 = \rho_{\pi\eta}$ and $\rho_2 = \rho_{KK}$, with ρ_i defined in Eq. (4). For the $f_0(980)$ the relevant coupling constants are $g_1 = g_{\pi\pi}$ and $g_2 = g_{KK}$, and the phase space terms are $\rho_1 = \rho_{\pi\pi}$

and $\rho_2 = \rho_{KK}$. The charged and neutral K channels are usually assumed to have the same coupling constant but different phase space factors, due to $m_{K^+} \neq m_{K^0}$; the result is

$$\rho_{KK} = \frac{1}{2}\left(\sqrt{1-\left(\frac{2m_{K^\pm}}{m_{KK}}\right)^2} + \sqrt{1-\left(\frac{2m_{K^0}}{m_{KK}}\right)^2}\right) . \qquad (18)$$

The effect of using this expression compared to using the averaged kaon masses is confined in the region very near threshold and is significant only in between the two kaon thresholds. If the coupling of a resonance to the channel opening nearby is strong, the Flatté parametrization shows a scaling invariance and does not allow for an extraction of the parameters individually, but only of ratios [27].

Further improvements:

The K–matrix described above usually allows one to get a proper fit of physical amplitudes and it is easy to deal with. However, it also has an important deficit: it violates constraints from analyticity — *e.g.*, ρ_{ii} has a pole at $s = 0$, and for unequal masses develops an unphysical cut. An analytic continuation of the amplitudes into the complex plane is not controlled, and typically the parameters of broad resonances come out wrong (see, *e.g.*, the minireview on scalar mesons). A method to improve the analytic properties was suggested in Refs. [25,28–30]. It basically amounts to replacing the phase-space factor $i\rho_i$ in Eqs. 9 and 14 with an analytic function that produces the identical imaginary part. In the simplest case of a channel with equal masses the expressions are

$$-\frac{\rho_i}{\pi}\log\left|\frac{1+\rho_i}{1-\rho_i}\right| , \ -\frac{2\rho_i}{\pi}\arctan\left(\frac{1}{\rho_i}\right) , \ -\frac{\rho_i}{\pi}\log\left|\frac{1+\rho_i}{1-\rho_i}\right| + i\rho_i$$

for $m^2 < 0$, $0 < m^2 < (m_a + m_b)^2$, and $(m_a + m_b)^2 < m^2$, respectively. Here $\rho_i = \sqrt{|1-(m_a+m_b)^2/m^2|}$ for all values of m^2, extending the expression of Eq. (4) into the regime below threshold. The more complicated expression for the case of different masses can be found, *e.g.*, in Ref. 29.

Branching Ratios from Dalitz Plot Fits: A fit to the Dalitz plot distribution using either a Breit-Wigner or a K-matrix formalism factorizes into a resonant contribution to the amplitude $\mathcal{M}_j$ and a complex coefficient, $a_j e^{i\delta_j}$, where a_j and δ_j are real. The definition of a rate of a single process, given a set of amplitudes a_j and phases δ_j, is the square of the relevant matrix element (see Eq. (1)). The "fit fraction" is usually defined as the integral over the Dalitz plot (m_{ab} vs. m_{bc}) of a single amplitude squared divided by the integral over the Dalitz plot of the square of the coherent sum of all amplitudes, or

$$\text{fit fraction}_j = \frac{\int \left|a_j e^{i\delta_j} \mathcal{M}_j\right|^2 dm_{ab}^2 dm_{bc}^2}{\int \left|\sum_k a_k e^{i\delta_k} \mathcal{M}_k\right|^2 dm_{ab}^2 dm_{bc}^2}, \tag{19}$$

where $\mathcal{M}_j$ is defined in Eq. (2) and described in Ref. 31. In general, the sum of the fit fractions for all components will not be unity due to interference.

When the K-matrix of Eq. (9) is used to describe a wave (*e.g.*, the $\pi\pi$ S-wave), then $\mathcal{M}_j$ refers to the entire wave. In this case, it may not be straightforward to separate $\mathcal{M}_j$ into a sum of individual resonances unless these are narrow and well separated.

Reconstruction Efficiency and Resolution: The efficiency for reconstructing an event as a function of position on the Dalitz plot is in general non-uniform. Typically, a Monte Carlo sample generated with a uniform distribution in phase space is used to determine the efficiency. The variation in efficiency across the Dalitz plot varies with experiment and decay mode. Most recent analyses utilize a full GEANT [32] detector simulation.

Finite detector resolution can usually be safely neglected, as most resonances are comparatively broad. Notable exceptions where detector resolution effects must be modeled are $\phi \to K^+K^-$, $\omega \to \pi^+\pi^-$, and $a_0 \to \eta\pi^0$. One approach is to convolve the resolution function in the Dalitz-plot variables m_{ab}^2 and m_{bc}^2 with the function that parametrizes the resonant amplitudes. In high-statistics data samples, resolution effects near the phase-space boundary typically contribute to a poor goodness of fit. The momenta of the final-state particles can be recalculated with a D or B mass constraint, which forces the kinematic boundaries of the Dalitz plot to be strictly respected.

If the three-body mass is not constrained, then the efficiency (and the parametrization of background) may also depend on the reconstructed mass.

Backgrounds: The contribution of background to D and B samples varies by experiment and final state. The background naturally falls into six categories: (1) Purely combinatoric background containing no resonances. (2) Combinatoric background containing intermediate resonances, such as a real K^* or ρ, plus additional random particles. (3) Final states containing identical particles as in $D^0 \to K_S^0\pi^0$ background to $D^0 \to \pi^+\pi^-\pi^0$ and $B \to D\pi$ background to $B \to K\pi\pi$. (4) Mistagged decays such as a real $\overline{D}^0$ or $\overline{B}^0$ incorrectly identified as a D^0 or B^0. (5) Particle misidentification of the decay products, such as $D^+ \to \pi^-\pi^+\pi^+$ or $D_s^+ \to K^-K^+\pi^+$ reconstructed as $D^+ \to K^-\pi^+\pi^+$. (6) Background from decays of charged pions or kaons in flight.

The contribution from combinatoric background with intermediate resonances is distinct from the resonances in the signal because the former do *not* interfere with the latter since they are not from true resonances. The usual identification tag of the initial particle as a D^0 or a $\overline{D}^0$ is the charge of the distinctive slow pion in the decay sequence $D^{*+} \to D^0\pi_s^+$ or $D^{*-} \to \overline{D}^0\pi_s^-$. Another possibility is the identification or "tagging" of one of the D mesons from $\psi(3770) \to D^0\overline{D}^0$, as is done for B mesons from $\Upsilon(4S)$. The mistagged background is subtle and may be mistakenly enumerated in the *signal* fraction determined by a D^0 mass fit. Mistagged decays contain true $\overline{D}^0$'s or $\overline{B}^0$'s and so the resonances in the mistagged sample exhibit interference on the Dalitz plot.

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